Geometric stacks.

We will now try to introduce a notion that cuts part of the category of stacks Stk into more reasonable objects.

Idea: start w/ a nice class of stacks, e.g. affine schemes or disjoint unions of such. and include objects obtained as quotients of groupoids in this category where the structure maps are "nice", e.g. étale or smooth.

The approach we will take tries to follow Lurie's thesis, for the reason that we hope to get its representability theorem in an "if and only if" form.

The definition is inductive, but the case $n=0$ is slightly different than the rest.

Def'n: A stack $\mathcal{X}$ is 0-geometric if

(i) ${}^{cl}\mathcal{X}$ is 0-truncated;
(ii) $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is affine representable;
(iii) $\exists$ $Z \to \mathcal{X}$ an étale and surjective morphism where Z is a disj. union of aff. schemes.

(such a Z is called an atlas).

Ex: Any scheme $Z \in Sch$ is 0-geometric, since ${}^{cl}Z$ is 0-truncated. (Talk 7)

- A morphism $f: \mathcal{X} \to \mathcal{Y}$ is 0-geometric if for every $S \to \mathcal{Y}$ w/ S aff. $\mathcal{X} \times_{\mathcal{Y}} S$ is 0-geometric.

- A morphism $\mathcal{X} \to S$ from a 0-geometric stack to an affine scheme is flat (resp. smooth, étale) if for some (equivalently for all) atlas $Z \to \mathcal{X}$, the composite $Z \to S$ is flat (resp. smooth, étale).
- A 0-geometric morphism $\mathcal{X} \to \mathcal{Y}$ is flat (resp. smooth, étale) if for every affine $S \to \mathcal{Y}$ $\mathcal{X} \times_{\mathcal{Y}} S \to S$ is such.

Suppose the notions above have been defined for all $k' \in \mathbb{N} = \{0,1,2,\dots\}$. $k' < k$.

Then we make the following:

Def'n: A stack $\mathcal{X}$ is k-geometric if:

(i) $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is $(k-1)$-geometric.
(ii) $\exists$ Z a $(k-1)$-geometric stack and
$f: Z \to \mathcal{X}$ a <u>smooth</u> & surjective map.
(called an atlas).

A morphism $f: \mathcal{X} \to \mathcal{Y}$ is <u>k-geometric</u> if for all $S \to \mathcal{Y}$ affine, $\mathcal{X} \times_{\mathcal{Y}} S$ is k-geometric.

~~A~~ A k-geometric morphism $f: \mathcal{X} \to \mathcal{Y}$ is <u>flat</u> (resp. <u>smooth</u>, <u>étale</u>) if for every affine $S \to \mathcal{Y}$ the morphism $\mathcal{X} \times_{\mathcal{Y}} S \to S$ is such, i.e. for an ~~cover of~~ atlas $Z \to \mathcal{X} \times_{\mathcal{Y}} S$ the composite $Z \to S$ is such.

<u>Rk:</u> Some places start w/ affine schemes for $n=0$, and consider smooth maps for all atlases.
There are some compatibilities to be checked about this definition. See [GR-I, Chapter 2, §4.2] for a nice discussion.

<u>Notice:</u> For $\mathcal{X}$ n-geometric & $f: Z \to \mathcal{X}$ an atlas. One has that f is an étale surjection.

Thus, <u>Cor:</u> $L(|Z^{\bullet}/\mathcal{X}|_{PStk}) \simeq |Z^{\bullet}/\mathcal{X}|_{Stk} \xrightarrow{\simeq} \mathcal{X}$.

This corollary starts to justify the intuition above, an n-geometric stack is obtained as the geometric realization of a groupoid object $Z^{\bullet}/\mathcal{X}$ in $(n-1)$-geometric stacks.

<u>Cor 2:</u> For any $\mathcal{X} \in Stk^{k\text{-geom.}}$, $^{\leq n}\mathcal{X}$ is $(n+k)$-truncated.

<u>Pf:</u> By induction on k. For $k=0$ (& $n=0$) it holds by defn.

Notice that the geometric realization of $(n-1)$-truncated objects in Spc is n-truncated. Since $^{\leq n}L$ takes n-truncated objects to n-truncated, it is enough to check each $^{\leq n}(Z^{\bullet i}/\mathcal{X})$ is $(n+k-1)$-truncated. but each $Z^{i}/\mathcal{X}) \in Stk^{(k-1)\text{-geom.}}$ & the result follows by induction ∎

The following result really justify the intuition in the initial discussion.

Prop: For $k \geq 1$. Let $\mathcal{X}^\bullet$ be a groupoid object in Stk. Suppose that.
- $\mathcal{X}^0, \mathcal{X}^1$ are k-geometric;
- $\mathcal{X}^1 \rightrightarrows \mathcal{X}^0$ are $(k-1)$-geometric and smooth;

Then $\mathcal{X} := |\mathcal{X}^\bullet|$ is k-geometric.

Pf: [See GR-II, Chapter 2, Prop. 4.3.6 for more details.] The idea is that it is enough to check

$$|\mathcal{X}^\bullet| \to |\mathcal{X}^\bullet| \times |\mathcal{X}^\bullet| \text{ is } (k-1)\text{-geometric.}$$

Since L takes k-geometric morphisms to k-geometric morphisms. (needs proof.)

Given $S \to |\mathcal{X}^\bullet| \times |\mathcal{X}^\bullet|$ this factors as $S \to \mathcal{X}^0 \times \mathcal{X}^0 \to |\mathcal{X}^\bullet| \times |\mathcal{X}^\bullet|$, so we have:

$$S \underset{|\mathcal{X}^\bullet|\times|\mathcal{X}^\bullet|}{\times} |\mathcal{X}^\bullet| \simeq S \underset{\mathcal{X}^0\times\mathcal{X}^0}{\times} \mathcal{X}^0\times\mathcal{X}^0 \underset{|\mathcal{X}^\bullet|\times|\mathcal{X}^\bullet|}{\times} |\mathcal{X}^\bullet| \simeq S \underset{\mathcal{X}^0\times\mathcal{X}^0}{\times} \left(\mathcal{X}^0 \underset{|\mathcal{X}^\bullet|}{\times} \mathcal{X}^0\right) \simeq S \underset{\mathcal{X}^0\times\mathcal{X}^0}{\times} \mathcal{X}^1.$$

and the claim follows from the assumption $\mathcal{X}^0 \times \mathcal{X}^0 - \mathcal{X}^1$ is $(k-1)$-geom.

One can also show $\mathcal{X}^0 \to |\mathcal{X}^\bullet|$ is smooth & surjective. w/ the same argument.

Finally, consider an atlas $Z^0 \xrightarrow{g} \mathcal{X}^0$, i.e. Z^0 is $(k-1)$-geom. and g is smooth & surjective. So the composite $Z^0 \to \mathcal{X}^0 \to |\mathcal{X}^\bullet|$ is an atlas of $|\mathcal{X}^\bullet|$. ∎

In particular, for $Z^1 \rightrightarrows Z^0$ smooth schematic morphisms and $Z^\bullet$ a groupoid objects extending) one has $L(|Z^\bullet|)$ is 1-geometric.

Examples: (i) $BG := |\cdots \substack{\to\\\to\\\to} G \times G \rightrightarrows G \rightrightarrows *|$ where G is a smooth group scheme. is 1-geometric.

Example (ii): Recall that for any $R \in CAlg$ we defined. $Vect(R)$ a stable ∞-category.

It is clear that this assembles to give a prestack:

$$Vect : Sch^{aff} \to Spc.$$
$$S \mapsto Vect(S)^{\simeq} := \text{underlying } \infty\text{-groupoid of } Vect(R), \text{ where } S = Spec(R).$$

One can check that Vect satisfies étale descent, this follows easily from the description of $Vect(R)$ as the smallest stable subcategory of $QCoh(R)$ containing R, $\oplus$'s and retracts.

We claim $Vect \simeq \coprod_{n \geq 0} Vect_n$, where $Vect_n(S) := \{ \mathcal{F} \in Vect(S) \mid \forall\, Spec\, \ell \to S$, ℓ a field, $\mathcal{F}|_{Spec\, \ell}$ has dim. $n \}$.

Moreover, $* \to Vect_n$ is an étale surjection.
$*(S) \mapsto \mathcal{O}_S^{\oplus n}$ (i.e. maps to the trivial rank n v.b.)

Indeed, in the classical case this follows from the def'n of vector bundles as locally trivial in the Zariski topology, so also in the étale topology.

The proof in the derived setting uses the classical fact + existences of homotopy inverses for projective modules. (See [SAG. Prop. 2.9.2.3].)

Thus, $|(*/Vect_n)^{\bullet}| \simeq Vect_n$.

Claim: $* \times_{Vect_n} * \simeq GL_n$, i.e. $* \times_{Vect_n} * (S) \simeq \{ \varphi \in Hom_{Vect(S)}(\mathcal{O}_S^{\oplus n}, \mathcal{O}_S^{\oplus n}) \mid h_0 \varphi \in Aut_{h_0 Vect(S)}(\mathcal{O}_S^{\oplus n}) \}$.

Assuming the claim, we obtain that $Vect_n$ is 1-geometric. Since
$GL_n(S) \simeq Spec(R[x_{ij}]_{1 \leq i,j \leq n}[det^{-1}])$. $S = Spec\, R$.

And in particular GL_n is affine.

Rk: ~~In the above def'ns of n-geometric one obtains the same class of objects~~

Toën has two results, concerning the class of n-geometric stacks.

(1) Any n-geometric stack satisfy flat and f.p. descent.

(2) If one required only f.f.p. descent & a f.f.p. atlas one automatically gets a smooth for $(n \geq 1)$ / étale for $(n=0)$ atlases.

Note this is similar to what happens in the usual def'n of algebraic spaces/stacks.

The following shows the consequences of geometricity to deformation theory.

<u>Prop:</u> Let $\mathcal{X}$ be an n-geometric stack. Then.

(i) $\mathbb{T}^*\mathcal{X} \in QCoh(\mathcal{X})^{\leq n}$ and $\mathcal{X}$ admits deformation thy.

(ii) Let $\mathcal{X} \to \mathcal{Y}_0$ be smooth where $\mathcal{Y}_0$ is 0-geom. then for any $x : S \to \mathcal{X}$ one has $T_x^*(\mathcal{X}/\mathcal{Y}_0) \in QCoh(S)^{\geq 0, \leq (n-1)}$.

<u>Idea of pf:</u> Consider $f : \mathcal{Z} \to \mathcal{X}$ st.

- $\mathcal{X}$ satisfies ét. descent
- f is an étale surjection.
- $\mathcal{Z}$ admits def. thy. we will prove that $\mathcal{X}$ has def. thy.
- f admits def. thy.
- f is formally smooth.

Consider $S_1' \coprod_{S_1} S_2 =: S_2'$ where $S_1 \to S_1'$ is a sq. zero.

we need to show:

given $S_2 \to \mathcal{X}$ $\quad \mathcal{X}(S_2') \times_{\mathcal{X}(S_2)} * \xrightarrow{\simeq} \mathcal{X}(S_1') \times_{\mathcal{X}(S_1)} *$

By étale descent the statement is étale local on S_2. So we assume $S_2 \to \mathcal{X}$ lifts to a map $S_2 \to \mathcal{Y}$. Thus we have:

$$\begin{tikzcd}
\left|(\mathcal{Z}/\mathcal{X})^{\circ}(S_2') \times_{(\mathcal{Z}/\mathcal{X})^{\circ}(S_2)} *\right| \arrow[r] \arrow[d] & \mathcal{X}(S_2') \times_{\mathcal{X}(S_2)} * \arrow[d] \\
\left|(\mathcal{Z}/\mathcal{X})^{\circ}(S_1') \times_{(\mathcal{Z}/\mathcal{X})^{\circ}(S_1)} *\right| \arrow[r] & \mathcal{X}(S_1') \times_{\mathcal{X}(S_1)} *
\end{tikzcd}$$

Notice the horizontal maps are monomorphisms, i.e.

$$\begin{tikzcd} S_2 \arrow[r] \arrow[d] & \mathcal{Z} \arrow[d] \\ S_2' \arrow[r, dashed] \arrow[ur, dashed] & \mathcal{X} \end{tikzcd}$$

So it is enough to prove they are surjective. This will follow from f being formally smooth, i.e. $\forall\, x : S \to \mathcal{Z}$ one has.

$$\operatorname{Hom}_{QCoh(S)}\left(T_x^*(\mathcal{Z}/\mathcal{X}), \mathcal{F}\right) \in Vect^{\leq 0}, \quad \text{for } \mathcal{F} \in QCoh(S)^{\heartsuit}. \quad (\star)$$

Given a diagram $\begin{tikzcd} S \arrow[r, "y"] \arrow[d] & \mathcal{Z} \arrow[d] \\ S' \arrow[r] \arrow[ur, dashed] & \mathcal{X} \end{tikzcd}$ we claim the dashed arrow exist.
w/ $S \hookrightarrow S'$ sq. zero.

Indeed, if $S \hookrightarrow S'$ is described by $T^*S \to \mathcal{F}$ w/ $\mathcal{F} \in QCoh(S)^{\leq -1}$, the space of such arrows is the space of null-homotopies of $T_y^*(\mathcal{Z}/\mathcal{X}) \to \mathcal{F}$, which is non-empty. b/c. of $(\star)$.

~~Thus, 2for~~ This gives by induction on n that any n-geometric stack has def'n. Thy. We now calculate the connectivity of their cotangent complex.

<u>$n=0$.</u> $f : \mathcal{Z} \to \mathcal{X}$ gives for any $x : S \to \mathcal{Z}$

$$T^*_{f\circ x}\mathcal{X} \to T^*_x\mathcal{Z} \to T^*_x(\mathcal{Z}/\mathcal{X}).$$

Since f is étale one has. $T^*_x(\mathcal{Z}/\mathcal{X}) \simeq 0$. Since $\mathcal{Z}$ is a disj. union of affines we have. $T^*_{f\circ x}\mathcal{X} \in QCoh(S)^{\leq 0}$.

<u>$n=1$</u> $T^*_{f\circ x}\mathcal{X} \to T^*_x\mathcal{Z} \to \underbrace{T^*_x(\mathcal{Z}/\mathcal{X})}_{\in\, QCoh(S)^{\heartsuit}}$ b/c. $\mathcal{Z} \to \mathcal{X}$ is smooth.

$\Rightarrow T^*_x\mathcal{X} \in QCoh(S)^{\leq 1}$.

By induction on n. one has $T^*_x\mathcal{X} \in QCoh(S)^{\leq n}$.